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Introduction to Homological Algebra

Lecture 2: 08/08/2007

§2.1. **Lemma** (Long exact sequence of Hom's). Suppose $\mathcal{D}$ is any triangulated category and

$$M \longrightarrow N \longrightarrow C \longrightarrow M[1]$$

is a distinguished triangle in $\mathcal{D}$. For any $L \in \mathcal{D}$, the sequence

$$\ldots \to \mathrm{Hom}_{\mathcal{D}}(L, C[-1]) \to \mathrm{Hom}_{\mathcal{D}}(L, M) \to$$
$$\to \mathrm{Hom}_{\mathcal{D}}(L, N) \to \mathrm{Hom}_{\mathcal{D}}(L, C) \to$$
$$\to \mathrm{Hom}_{\mathcal{D}}(L, M[1]) \to \mathrm{Hom}_{\mathcal{D}}(L, N[1]) \to \ldots$$

is exact.

Proof: Exercise (note that by the rotation invariance, it is enough to check exactness at just one term).

§2.2. **Lemma** (also an exercise): The direct sum of two triangles is distinguished if and only if each of them is distinguished.

§2.3. Definition. A *triangulated functor* between triangulated categories $\mathcal{D}_1$ and $\mathcal{D}_2$ is an additive functor

$$F : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$$

equipped with natural identifications

$$F(M[1]) \xrightarrow{\simeq} F(M)[1] \quad \forall M \in \mathcal{D}_1 \qquad (*)$$

such that F takes distinguished triangles in $\mathcal{D}_1$ to distinguished triangles in $\mathcal{D}_2$.

A *morphism* of triangulated functors is a morphism of the underlying additive functors which is compatible with the identifications $(*)$.

§2.4. Definition. If $\mathcal{D}$ is a triangulated category and $\mathcal{A}$ is an abelian category, an additive functor $H : \mathcal{D} \longrightarrow \mathcal{A}$ is said to be *cohomological* if it transforms distinguished triangles into long exact sequences.

§2.5. Suppose $\mathcal{D}$ is a triangulated category and $\{M_\alpha\}$ is a collection of objects of $\mathcal{D}$. The *triangulated category generated by* $\{M_\alpha\}$ is the smallest strictly full triangulated subcategory of $\mathcal{D}$ containing each of the objects M_α.

§2.6. Exercises. (1) Suppose $F : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ is a triangulated functor between triangulated categories, and $\{M_\alpha\}$ is a collection of objects of $\mathcal{D}_1$ which generate $\mathcal{D}_1$.

Assume that (a) F induces isomorphisms

$$\mathrm{Hom}^n(M_\alpha, M_\beta) \xrightarrow{\simeq} \mathrm{Hom}^n(F(M_\alpha), F(M_\beta)) \quad \forall \alpha, \beta$$

(b) the objects $\{F(M_\alpha)\}$ generate $\mathcal{D}_2$.

Show that (a) $\Rightarrow$ F is fully faithful

(a) + (b) $\Rightarrow$ F is an equivalence.

(2) Prove that if a triangulated functor has an adjoint, then this adjoint is automatically triangulated.

§2.7. Verdier's quotient construction

Suppose we have a triangulated category $\mathcal{D}$ and a family $\{M_\alpha\}$ of objects of $\mathcal{D}$. Consider triangulated functors from $\mathcal{D}$ to other triangulated categories which kill each of the objects M_α.

Does there exist a universal such functor, $\mathcal{D} \longrightarrow \mathcal{D}/\{M_\alpha\}$. We will see that it does, modulo some set-theoretical issues (e.g., if $\mathcal{D}$ is essentially small, there is no problem).

<u>Remark</u>: To solve the problem above, we may assume that the M_α's are objects of a strictly full triangulated subcategory $\mathcal{V} \subset \mathcal{D}$.
(Indeed, any triangulated functor $\mathcal{D} \to \mathcal{D}'$ which kills every M_α also kills the triangulated subcategory generated by $\{M_\alpha\}$.)

§2.8. Now consider a triangulated category $\mathcal{D}$ and a strictly full triangulated subcategory $\mathcal{V} \subset \mathcal{D}$. We want to give a construction of the Verdier quotient $\mathcal{D}/\mathcal{V}$.

<u>Remark</u>: We will see that every object of $\mathcal{D}$ which is killed by the quotient functor $\mathcal{D} \longrightarrow \mathcal{D}/\mathcal{V}$ is in fact a direct summand of an object of $\mathcal{V}$.

§2.9. <u>Def</u>: A <u>$\mathcal{V}$-quasi-isomorphism</u> is a morphism $M \xrightarrow{u} N$ in $\mathcal{D}$ such that $\mathrm{Cone}(u) \in \mathcal{V}$.

[<u>Remark</u>. $\mathrm{Cone}(u)$ is determined uniquely up to a <u>non-unique</u> isomorphism. It is easy to check that u is an isomorphism in $\mathcal{D}$ if and only if $\mathrm{Cone}(u) = 0$ in $\mathcal{D}$.]

Exercise. Show that if f, g are composable arrows in $\mathcal{D}$ and two out of the three arrows $f, g, g\circ f$ are $\mathcal{V}$-quasi-isomorphisms, then so is the third one. (*Hint*: use the octahedron axiom.)

We want to define $\mathcal{D}/\mathcal{V}$ by formally inverting all the $\mathcal{V}$-quasi-isomorphisms in $\mathcal{D}$. In general, such an operation may lead to a very ugly result. Here, however, there exists a nice and simple construction of the localization "in one step".

$\boxed{\S 2.10.}$ Fix $M \in \mathcal{D}$. Write Q/M for the full subcategory of the category $\mathcal{D}/M$ of objects of $\mathcal{D}$ over M formed by the $\mathcal{V}$-quasi-isomorphisms $N \longrightarrow M$.

Lemma: $(Q/M)^{op}$ is a directed category.

$\boxed{\S 2.11.}$ *Digression.* A category $\mathcal{L}$ is **directed** if it satisfies the following three properties:

(1) For any pair of objects $I_1, I_2 \in \mathcal{L}$, there exists a diagram in $\mathcal{L}$ of the form

$$\begin{array}{l} I_1 \longrightarrow I_3 \\ I_2 \nearrow \end{array}$$

(2) Any diagram in $\mathcal{I}$ of the form

$$\begin{tikzcd} & I_2 \\ I_1 \arrow[ur] \arrow[dr] & \\ & I_3 \end{tikzcd}$$

can be completed to a commutative square

$$\begin{tikzcd} & I_2 \arrow[dr, dashed] & \\ I_1 \arrow[ur] \arrow[dr] & & I_4 \\ & I_3 \arrow[ur, dashed] & \end{tikzcd}$$

(3) Given any parallel pair of morphisms $I_1 \underset{g}{\overset{f}{\rightrightarrows}} I_2$ in $\mathcal{I}$, there exists a morphism $I_2 \xrightarrow{h} I_3$ in $\mathcal{I}$ with $h \circ f = h \circ g$.

<u>Remark</u>: This is a useful notion for the following reason. Suppose $\mathcal{I}$ is a small category. For any functor $F : \mathcal{I} \longrightarrow \mathcal{A}b$, one has the inductive limit $\varinjlim_{\mathcal{I}} F \in \mathcal{A}b$, and we get a functor

$$\varinjlim_{\mathcal{I}} : \mathrm{Funct}(\mathcal{I}, \mathcal{A}b) \longrightarrow \mathcal{A}b$$

Now if $\mathcal{I}$ is a directed category, then the functor $\varinjlim_{\mathcal{I}}$ is exact.

$\boxed{\S 2.12.}$ Dual statement to Lemma 2.10: for any $M \in \mathcal{D}$ as above, the category $Q \backslash M$ of $\mathcal{P}$-quasi-isomorphisms $M \longrightarrow M'$ is also directed.

$\boxed{\S 2.13.}$ The proofs of (2.10) and the dual statement (2.12) are very easy.

<u>Remark:</u> By the exercise in §2.9, all morphisms in the categories $Q \backslash M$ and Q/M are automatically $\mathcal{P}$-quasi-isomorphisms in $\mathcal{D}$.

For example, how do we prove property (2) for the category $(Q/M)^{op}$?

$$\begin{array}{l} M_2 \xrightarrow[\alpha]{qis} M_1 \xrightarrow{qis} M \\ M_3 \xrightarrow[qis]{\beta} M_1 \end{array}$$

$\rightsquigarrow$ consider $\alpha + \beta : M_2 \oplus M_3 \longrightarrow M_1$ and complete to a distinguished triangle

$$M_4 \xrightarrow{(\gamma, \delta)} M_2 \oplus M_3 \xrightarrow{\alpha+\beta} M_1 \longrightarrow M_4[1]$$

Check that γ, δ are $\mathcal{P}$-quasi-isomorphisms.

We get a commutative diagram

$$\begin{tikzcd} & M_2 \arrow[dr, "\alpha", "qis"'] & & \\ M_4 \arrow[ur, "\gamma", "qis"'] \arrow[dr, "qis", "\delta"'] & & M_1 \arrow[r] & M \\ & M_3 \arrow[ur, "qis", "\beta"'] & & \end{tikzcd}$$

For property (3), consider

$$M_2 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M_1 \xrightarrow{qis} M$$

Now

$$\begin{tikzcd} & & V \in \mathcal{P} \arrow[dr] & & \\ & M_2 \arrow[ur] \arrow[rr, "\alpha - \beta"] & & M_1 \arrow[dr] & \\ M_3 \arrow[ur] & & & & M \end{tikzcd}$$

complete to a distinguished triangle

get this using the long exact sequence of Homs

complete to a dist. triangle

$\mathcal{P}$-quasi-isom by construction

<u>Remark.</u> (2.12) follows from (2.10) formally by considering $\mathcal{D}^{op}$.

$\boxed{\S 2.14.}$ Now we are ready to define the Verdier quotient category $\mathcal{D}/\mathcal{P}$.

$$\mathrm{Objects}(\mathcal{D}/\mathcal{P}) = \mathrm{Objects}(\mathcal{D})$$

$$\mathrm{Hom}_{\mathcal{D}/\mathcal{P}}(M, N) = \varinjlim_{M' \in (Q/M)^{op}} \mathrm{Hom}_{\mathcal{D}}(M', N)$$

$$\cong \varinjlim_{N' \in Q \backslash N} \mathrm{Hom}_{\mathcal{D}}(M, N')$$

we need to check that there is a canonical isomorphism like this

Key thing one needs to check: given one of the commutative triangles in a diagram

$$\begin{tikzcd}
& M' \arrow[dl, "qis"'] \arrow[dr] & \\
M \arrow[rr] \arrow[dr] & & N \arrow[dl, "qis"] \\
& N' &
\end{tikzcd}$$

one can find a second triangle which makes the whole diagram commute.

$\longrightarrow$ For all the details (definition of the composition and addition of morphisms in $\mathcal{D}/\mathcal{V}$, and the triangulated structure on $\mathcal{D}/\mathcal{V}$, etc.), see:

- Verdier's thesis (one of the Astérisque volumes), or
- Gelfand and Manin, "Methods of Homological Algebra", chapters III and IV.

$\boxed{\S 2.15.}$ __Lemma:__ If an object $M \in \mathcal{D}$ is killed by the quotient functor $\mathcal{D} \longrightarrow \mathcal{D}/\mathcal{V}$, then M is a direct summand of an object of $\mathcal{V}$.

<u>Proof</u>: If M is killed in $\mathcal{D}/\mathcal{P}$, then $\exists\, M'$ such that $M' \xrightarrow{0} M$ is a $\mathcal{P}$-quasi-isomorphism. This implies that M is a direct summand of an object of $\mathcal{P}$. //.