

Introduction to Homological Algebra

Lecture 2: 08/08/2007

§2.1. Lemma (Long exact sequence of Hom's). Suppose \mathcal{D} is any triangulated category and

$$M \rightarrow N \rightarrow C \rightarrow M[1]$$

is a distinguished triangle in \mathcal{D} . For any $L \in \mathcal{D}$, the sequence

$$\begin{aligned} \dots &\rightarrow \text{Hom}_{\mathcal{D}}(L, C[-1]) \rightarrow \text{Hom}_{\mathcal{D}}(L, M) \rightarrow \\ &\rightarrow \text{Hom}_{\mathcal{D}}(L, N) \rightarrow \text{Hom}_{\mathcal{D}}(L, C) \rightarrow \\ &\rightarrow \text{Hom}_{\mathcal{D}}(L, M[1]) \rightarrow \text{Hom}_{\mathcal{D}}(L, N[1]) \rightarrow \dots \end{aligned}$$

is exact.

Proof: Exercise (note that by the rotation invariance, it is enough to check exactness at just one term).

§2.2. Lemma (also an exercise): The direct sum of two triangles is distinguished if and only if each of them is distinguished.

§2.3. Definition. A triangulated functor between triangulated categories \mathcal{D}_1 and \mathcal{D}_2 is an additive functor

$$F: \mathcal{D}_1 \longrightarrow \mathcal{D}_2$$

equipped with natural identifications

$$F(M[1]) \cong F(M)[1] \quad \forall M \in \mathcal{D}_1 \quad (*)$$

such that F takes distinguished triangles in \mathcal{D}_1 to distinguished triangles in \mathcal{D}_2 .

A morphism of triangulated functors is a morphism of the underlying additive functors which is compatible with the identifications (*).

§2.4. Definition. If \mathcal{D} is a triangulated category and \mathcal{A} is an abelian category, an additive functor $H: \mathcal{D} \rightarrow \mathcal{A}$ is said to be cohomological if it transforms distinguished triangles into long exact sequences.

§2.5. Suppose \mathcal{D} is a triangulated category and $\{M_\alpha\}$ is a collection of objects of \mathcal{D} . The triangulated category generated by $\{M_\alpha\}$ is the smallest strictly full triangulated subcategory of \mathcal{D} containing each of the objects M_α .

§2.6. Exercises. (1) Suppose $F: \mathcal{D}_1 \rightarrow \mathcal{D}_2$ is a triangulated functor between triangulated categories, and $\{M_\alpha\}$ is a collection of objects of \mathcal{D}_1 which generate \mathcal{D}_1 .

Assume that (a) F induces isomorphisms

$$\text{Hom}^n(M_\alpha, M_\beta) \xrightarrow{\cong} \text{Hom}^n(F(M_\alpha), F(M_\beta)) \quad \forall \alpha, \beta$$

(b) the objects $\{F(M_\alpha)\}$ generate \mathcal{D}_2 .

Show that (a) \Rightarrow F is fully faithful

(a) + (b) \Rightarrow F is an equivalence.

(2) Prove that if a triangulated functor has an adjoint, then this adjoint is automatically triangulated.

§2.7. Verdier's quotient construction

Suppose we have a triangulated category \mathcal{D} and a family $\{M_\alpha\}$ of objects of \mathcal{D} . Consider triangulated functors from \mathcal{D} to other triangulated categories which kill each of the objects M_α .

Does there exist a universal such functor, $\mathcal{D} \rightarrow \mathcal{D}/\{M_\alpha\}$. We will see that it does, modulo some set-theoretical issues (e.g., if \mathcal{D} is essentially small, there is no problem).

Remark: To solve the problem above, we may assume that the M_α 's are objects of a strictly full triangulated subcategory $\mathcal{T} \subset \mathcal{D}$.

(Indeed, any triangulated functor $\mathcal{D} \rightarrow \mathcal{D}'$ which kills every M_α also kills the triangulated subcategory generated by $\{M_\alpha\}$.)

§2.8. Now consider a triangulated category \mathcal{D} and a strictly full triangulated subcategory $\mathcal{T} \subset \mathcal{D}$. We want to give a construction of the Verdier quotient \mathcal{D}/\mathcal{T} .

Remark: We will see that every object of \mathcal{D} which is killed by the quotient functor $\mathcal{D} \rightarrow \mathcal{D}/\mathcal{T}$ is in fact a direct summand of an object of \mathcal{T} .

§2.9. Def: A \mathcal{T} -quasi-isomorphism is a morphism $M \xrightarrow{u} N$ in \mathcal{D} such that $\text{Cone}(u) \in \mathcal{T}$.

Remark. $\text{Cone}(u)$ is determined uniquely up to a non-unique isomorphism. It is easy to check that u is an isomorphism in \mathcal{D} if and only if $\text{Cone}(u) = 0$ in \mathcal{D} .

Exercise. Show that if f, g are composable arrows in \mathcal{D} and two out of the three arrows $f, g, g \circ f$ are \mathcal{V} -quasi-isomorphisms, then so is the third one. (Hint: use the octahedron axiom.)

We want to define \mathcal{D}/\mathcal{V} by formally inverting all the \mathcal{V} -quasi-isomorphisms in \mathcal{D} . In general, such an operation may lead to a very ugly result. Here, however, there exists a nice and simple construction of the localization "in one step".

§ 2.10. Fix $M \in \mathcal{D}$. Write \mathcal{Q}/M for the full subcategory of the category \mathcal{D}/M of objects of \mathcal{D} over M formed by the \mathcal{V} -quasi-isomorphisms $N \rightarrow M$.

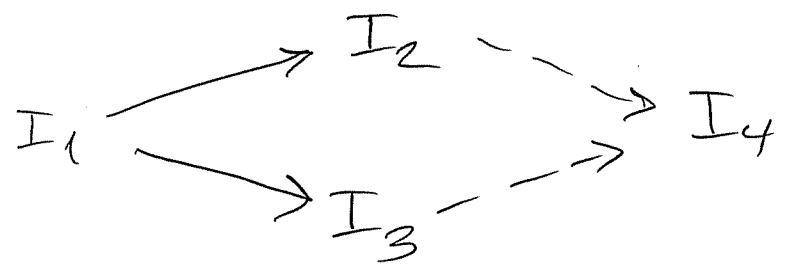
Lemma: $(\mathcal{Q}/M)^{op}$ is a directed category.

§ 2.11. Digression A category \mathcal{L} is directed if it satisfies the following three properties:

(1) For any pair of objects $I_1, I_2 \in \mathcal{L}$, there exists a diagram in \mathcal{L} of the form



(2) Any diagram in \mathcal{L} of the form $I_1 \begin{matrix} \longrightarrow & I_2 \\ & \searrow & \longrightarrow \\ & & I_3 \end{matrix}$ can be completed to a commutative square



(3) Given any parallel pair of morphisms $I_1 \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} I_2$ in \mathcal{L} , there exists a morphism $I_2 \xrightarrow{h} I_3$ in \mathcal{L} with $h \circ f = h \circ g$.

Remark: This is a useful notion for the following reason. Suppose \mathcal{L} is a small category. For any functor $F: \mathcal{L} \rightarrow \mathcal{A}\mathcal{B}$, one has the inductive limit $\varinjlim_{\mathcal{L}} F \in \mathcal{A}\mathcal{B}$, and we get a functor $\varinjlim_{\mathcal{L}}: \text{Funct}(\mathcal{L}, \mathcal{A}\mathcal{B}) \rightarrow \mathcal{A}\mathcal{B}$

Now if \mathcal{L} is a directed category, then the functor $\varinjlim_{\mathcal{L}}$ is exact.

§2.12. Dual statement to Lemma 2.10:

For any $M \in \mathcal{D}$ as above, the category Q/M of \mathcal{P} -quasi-isomorphisms $M \rightarrow M'$ is also directed.

§2.13. The proofs of (2.10) and the dual statement (2.12) are very easy.

Remark: By the exercise in §2.9, all morphisms in the categories Q/M and Q/M are automatically \mathcal{P} -quasi-isomorphisms in \mathcal{D} .

For example, how do we prove property (2) for the category $(Q/M)^{op}$?

$$\begin{array}{ccc}
 M_2 & \xrightarrow[\alpha]{qis} & M_1 \xrightarrow{qis} M \\
 & \nearrow \beta & \\
 M_3 & \xrightarrow[qis]{\beta} &
 \end{array}$$

\implies consider $\alpha + \beta : M_2 \oplus M_3 \rightarrow M_1$ and complete to a distinguished triangle

$$M_4 \xrightarrow{(\gamma, \delta)} M_2 \oplus M_3 \xrightarrow{\alpha + \beta} M_1 \rightarrow M_4[1]$$

check that γ, δ are \mathcal{P} -quasi-isomorphisms.

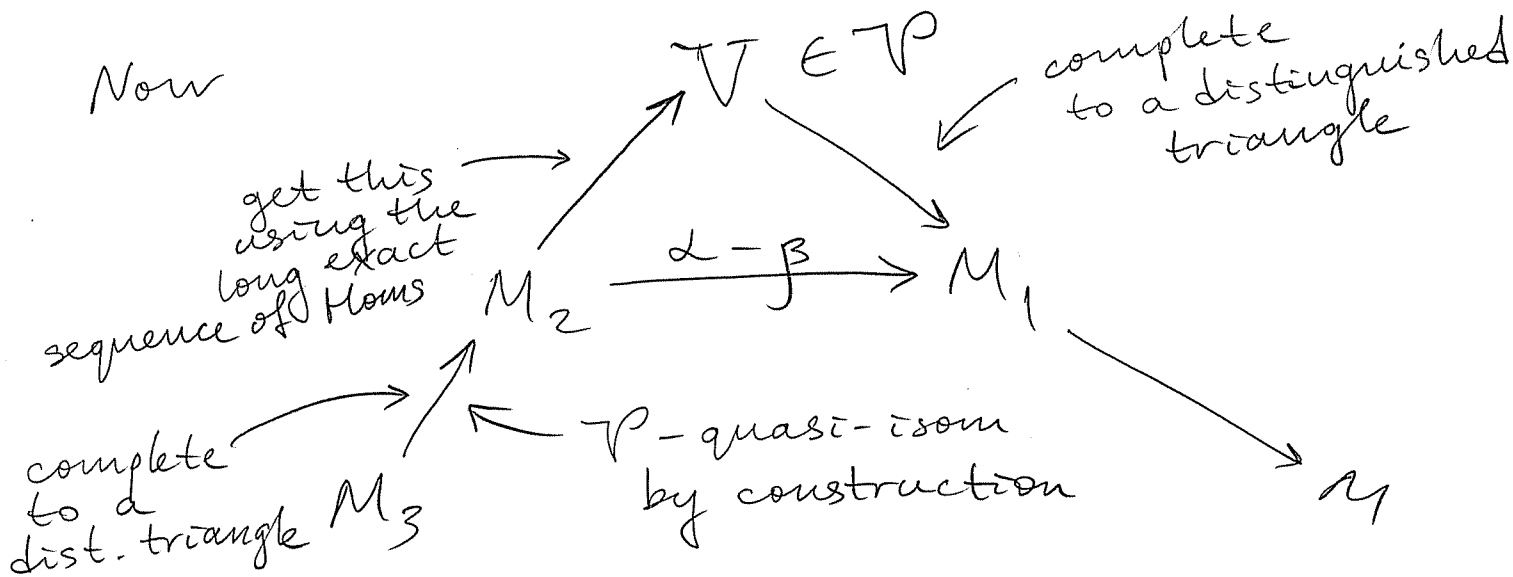
We get a commutative diagram

$$\begin{array}{ccccc}
 & & M_2 & \xrightarrow[\alpha]{qis} & M_1 & \longrightarrow & M \\
 & \nearrow \gamma & & & & & \\
 M_4 & & & & & & \\
 & \searrow \delta & & & & & \\
 & & M_3 & \xrightarrow[\beta]{qis} & & &
 \end{array}$$

For property (3), consider

$$M_2 \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} M_1 \xrightarrow{q \text{ is}} M$$

Now



Remark, (2.12) follows from (2.10) formally by considering \mathcal{D}^{op} .

§2.14. Now we are ready to define the Verdier quotient category \mathcal{D}/\mathcal{T} .

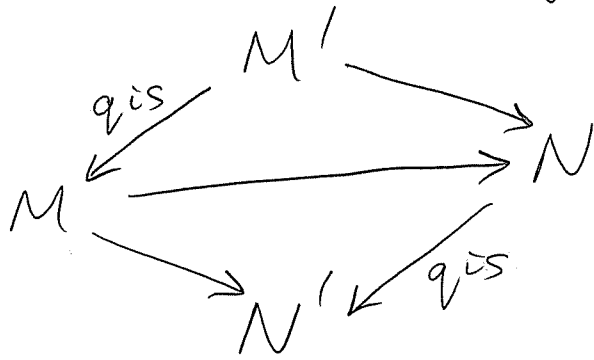
Objects $(\mathcal{D}/\mathcal{T}) = \text{Objects } (\mathcal{D})$

$$\text{Hom}_{\mathcal{D}/\mathcal{T}}(M, N) = \varinjlim_{M' \in (\mathcal{Q}/M)^{\text{op}}} \text{Hom}_{\mathcal{D}}(M', N)$$

$$\cong \varinjlim_{N' \in \mathcal{Q} \setminus N} \text{Hom}_{\mathcal{D}}(M, N')$$

we need to check that there is a canonical isomorphism like this

Key thing one needs to check:
given one of the commutative triangles in a diagram



one can find a second triangle which makes the whole diagram commute.

→ For all the details (definition of the composition and addition of morphisms in \mathcal{D}/\mathcal{V} , and the triangulated structure on \mathcal{D}/\mathcal{V} , etc.), see:

- Verdier's thesis (one of the Astérisque volumes), or

- Gelfand and Manin, "Methods of Homological Algebra", chapters III and IV.

§2.15. Lemma: If an object $M \in \mathcal{D}$ is killed by the quotient functor $\mathcal{D} \rightarrow \mathcal{D}/\mathcal{V}$, then M is a direct summand of an object of \mathcal{V} .

Proof: If M is killed in \mathcal{D}/\mathcal{V} ,
then $\exists M'$ such that $M' \xrightarrow{\circ} M$ is a
 \mathcal{V} -quasi-isomorphism. This implies
that M is a direct summand of
an object of \mathcal{V} . //